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NOTE

EDGE-COLORING CLIQUES WITH MANY COLORS ON SUBCLIQUES

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The second author constructed a coloring of $E(K_n)$ with $e^{O(\sqrt{\log n})}$ colors in which the edges of every copy of K_4 together receive at least 3 colors. We prove that this construction also has the property that at least $2\lceil \lg p\rceil - 2$ colors appear on the edges of every copy of K_p for $p \geq 5$.

1. Background

A (p,q)-*coloring of* K_n is a coloring of $E(K_n)$ in which the edges of every copy of $K_p \subseteq K_n$ together receive at least q colors. Erdős [2] asked for the minimum number of colors in a (p,q)-coloring of K_n and called this $f(n,p,q)$. This question generalizes the Ramsey problem for multicolorings, since $(p,2)$-colorings are just colorings with no monochromatic K_p. Results for the classical Ramsey problem yield corresponding results for $f(n,p,2)$; in particular, for $p=3$, bounds from [1] and [4] imply that $c\frac{\log n}{\log\log n} < f(n,3,2) < c' \log n$, where c and c' are constants.

Although $f(n,p,q)$ was first studied by Elekes, Erdős, and Füredi (as described in Section 9 of [2]), its growth rate was more thoroughly investigated by Erdős and Gyárfás [3]. They considered the case when p is fixed and $n \to \infty$, and determined the smallest value of q for which $f(n,p,q)$ is linear in n, and the smallest value of q for which $f(n,p,q)$ is quadratic in n. They also posed the following general problem:

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Problem. **[Erdős-Gyárfás [3]]** *Suppose that* $p \geq 4$ *is fixed. Decide whether* $f(n,p,p-1) \geq cn^\epsilon$ *for some* c *and* ϵ *depending only on* p*. More generally, determine the smallest value of* q *(in terms of* p*) for which* $f(n,p,q) \geq cn^\epsilon$*.*

In [3], it was observed that $f(n,p,\lceil \lg p \rceil) \leq \lceil \lg n \rceil$, and thus the threshold for q in the problem is at least $\lceil \lg p \rceil + 1$ (here lg denotes $\log_2$). The smallest special case that merits attention in this context is $f(n,4,3)$, which was shown to be $O(\sqrt{n})$ by probabilistic methods in [3]. In [5], this bound was improved by an explicit construction of a $(4,3)$-coloring of K_n with $e^{O(\sqrt{\log n})}$ colors, thereby answering the first part of the problem in the negative for $p=4$ (note that $e^{\sqrt{\log n}} = n^{1/\sqrt{\log n}}$). In this paper, we prove that the above construction is also a $(5,4)$-coloring of K_n, thus extending the negative result to $p=5$. More generally, we prove that for fixed $p>4$, this construction is a $(p, 2\lceil \lg p \rceil - 2)$-coloring of K_n. Hence the threshold for q in the problem is much greater than $\lceil \lg p \rceil + 1$. Our main result is

Theorem 1. *Suppose that* $p \geq 5$ *is fixed and that* $q = 2\lceil \lg p \rceil - 5 + \left\lceil \frac{4p}{2^{\lceil \lg p \rceil}} \right\rceil$*. As* $n \to \infty$*,*

$$f(n,p,q) < e^{\sqrt{4 \log 2 \log n}\ (1+o(1))}.$$

Since q is either $2\lceil \lg p \rceil - 1$ or $2\lceil \lg p \rceil - 2$, Theorem 1 shows that $f(n,p,2\lceil \lg p \rceil - 2)$ grows slower than any power of n for $p \geq 5$. Even for q as small as $\lceil \lg p \rceil + 1$, the previous best upper bound was as large as $c_p n^{\epsilon_p}$.

2. The "subset ranking" construction

In this section, we describe the construction in [5]. Because the coloring arises from the subsets of a specified set and uses the notion of ranking these subsets, we call it the SUBSET RANKING (SR) coloring. For $m>0$, let $[m] = \{1,\ldots,m\}$, and let $[0] = \emptyset$. For $t \leq m$, we write $\binom{[m]}{t}$ for the family of all subsets of $[m]$ with size t. The *symmetric difference* of the sets A and B is $A \triangle B = (A-B) \cup (B-A)$.

The SR coloring.
Let G be the complete graph with vertex set $\binom{[m]}{t}$. For each t-set T of $[m]$, rank the $2^t - 1$ proper subsets of T according to some linear order. Given distinct subsets A, B of $[m]$, let $S(A,B)$ denote the member of $\{A,B\}$ that contains the minimum element of $A \triangle B$. Color the edge AB with the two dimensional vector $c(AB) = (c_0(AB), c_1(AB))$, where $c_0(AB)$ is the

minimum element of $A \triangle B$, and $c_1(AB)$ is the rank of $A \cap B$ in the linear order associated with the proper subsets of $S(A,B)$. ∎

It is readily observed that the number of colors in the SR coloring is at most $(2^t - 1)(m-1)$, and that monochromatic triangles are forbidden. By choosing m and t appropriately with respect to n, one can show that the SR coloring restricted to K_n uses $e^{\sqrt{c \log n}(1+o(1))}$ colors, where $c = 4\log 2$. It can also be shown that this is a $(4,3)$-coloring (see [5] for the details).

3. Many colors on every K_p

Lemma 2. *Let $p_q = 2^{q/2} + 1$ when q is even, and let $p_q = 3 \cdot 2^{(q-3)/2} + 1$ when q is odd. For $p > 1$, let $Q(p) = 2\lceil \lg p \rceil - 5 + \left\lceil \frac{4p}{2^{\lceil \lg p \rceil}} \right\rceil$. Then for $q > 1$, the smallest integer x satisfying $Q(x) = q$ is p_q.*

Proof. Short calculations yield $Q(p_q) = q$ and $Q(p_q - 1) = q - 1$ for $q > 1$. Since $Q(p)$ is a non-decreasing function of p for $p > 1$, the result follows. ∎

Proof of Theorem 1. The SR coloring restricted to K_n is a $(3,2)$-coloring, and in [5] it is also shown to be a $(4,3)$-coloring with $e^{\sqrt{c \log n}(1+o(1))}$ colors, where $c = 4\log 2$. By Lemma 2 and the fact that $f(n,p,q)$ is a non-increasing function of p, it suffices to show that the SR coloring is also a (p_q, q)-coloring for $q > 3$. We proceed by induction on q. Suppose that the result holds for both $q-2$ and $q-1$, and consider the SR coloring restricted to a copy H of K_{p_q}.

Let $i = \min\{c_0(e) : e \in E(H)\}$ and assume $e = AB$ realizes this minimum. Without loss of generality we may assume $i \notin A$ and $i \in B$. Set $s = \lceil p_q/2 \rceil = p_{q-2}$. Either at least half of the sets representing the vertices of K_{p_q} contain i or at least half of the sets representing the vertices of K_{p_q} do not contain i. Thus there is a set $S = \{X_1, \ldots, X_s\} \subseteq V(H)$ such that either $i \in \cap_j X_j$ or $i \notin \cup_j X_j$. Now set $Y = A$ in the first case and $Y = B$ in the second case. Observe that $i \notin Y$ in the first case and $i \in Y$ in the second case. If Y is the only vertex in $V(H)$ with this property, then the colors on the edges of $H - \{Y\}$ all differ from i in their first coordinate. Since $|V(H - \{Y\})| = p_q - 1 \geq p_{q-1}$, the induction hypothesis implies that at least $q-1$ colors appear on $E(H - \{Y\})$. Together with the color on any edge incident to Y, this yields the required q colors on $E(H)$.

Thus we may assume that there is another vertex $Y' \in V(H)$ with $i \notin Y \triangle Y'$. Let $T = \{X_j Y : j \in [s]\} \cup \{X_j Y' : j \in [s]\}$. Since $|S| = p_{q-2}$, the induction hypothesis implies that at least $q-2$ colors appear in $E(H[S])$.

Because $c_0(e) = i$ for $e \in T$, these colors are distinct from the colors on edges in T. This yields the required q colors on $E(H)$ unless the edges in T are monochromatic, which we may henceforth assume. Since $i \notin Y \triangle Y'$, we conclude that $c(YY')$ differs from the color on T. To complete the proof, it suffices to show that $c(YY')$ is absent in $E(H[S])$.

Case 1. $i \in (\bigcap_j X_j) - (Y \cup Y')$. Because T is monochromatic, $c_1(X_jY) = c_1(X_jY')$. Since the rank of a subset in a set identifies the subset, $X_j \cap Y = X_j \cap Y'$ for all $j \in [s]$. Since $c_0(YY') \in Y \triangle Y'$, we conclude that $c_0(YY') \notin \bigcup_j X_j$, and thus $c(YY')$ is absent in $E(H[S])$.

Case 2. $i \in (Y \cap Y') - (\bigcup_j X_j)$. In this case, $X_j \cap Y = X_{j'} \cap Y$ and $X_j \cap Y' = X_{j'} \cap Y'$ for $j \neq j'$. Since $c_0(X_jX_{j'}) \in X_j \triangle X_{j'}$, we conclude that $c_0(X_jX_{j'}) \notin (Y \cup Y')$. Since $c_0(YY') \in Y \cup Y'$, the color $c(YY')$ is absent in $E(H[S])$. ∎

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